Linear Diophantine Equations (LDEs)

Definition 1

An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

with $a_1, a_2, \ldots, a_n, b$ integers, is called a linear Diophantine equation (LDE).

Theorem 2

The LDE

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

has a solution $x_1, ..., x_n \in \mathbb{Z}$ *if and only if* $gcd(a_1, a_2, \ldots, a_n)|b$

Quadratic Diophantine Equations (QDEs)

Definition 3

An equation of the form

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j = b \tag{2}$$

with a_{ij}, b integers, is called a quadratic Diophantine equation (QDE).

Example 4 (Pythagorean Equations)

The equation

$$x^2 + y^2 = z^2$$

is a QDE. Any solution (x, y, z) of this equation for integers x, y, z is called a Pythagorean triple.

Pythagorean Equations

Consider the Pythagorean equation:

$$x^2 + y^2 = z^2. \tag{3}$$

- A solution (x_0, y_0, z_0) of Eq. (3) where x_0, y_0, z_0 are pairwise relatively prime is called a primitive solution.
- If (x_0, y_0, z_0) is a solution of Eq. (3) then so are

$$(\pm x_0, \pm y_0, \pm z_0) \text{ and } (kx_0, ky_0, kz_0).$$

- Therefore we are most interested in solutions (x, y, z) of Eq. (3) with all components positive.

Pythagorean Equations

Theorem 5

Any primitive solution of

$$x^2 + y^2 = z^2$$

is of the form

$$x = m^2 - n^2, y = 2mn, z = m^2 + n^2 \tag{4}$$

Where $m, n \geq 1$ *are relatively prime positive integers.*

Pell's Equation

Definition 6

Pell's equation has the form

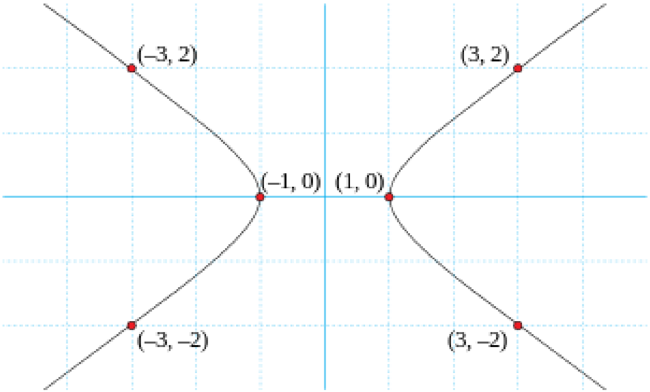
$$x^2 - dy^2 = 1 \tag{5}$$

where d not a perfect square.

Definition 7

We say that (x_0, y_0) is a fundamental solution of Pell's equation if x_0, y_0 are positive integers that are minimal amongst all solutions.

The Graph of Pell's Equation



$$x^2 - 2y^2 = 1$$

The equation has the fundamental solution $(x_0, y_0) = (3,2)$.

Pell's Equation

Theorem 8

Pell's equation has infinitely many solutions. Given the solution (x_0, y_0) the solution (x_{n+1}, y_{n+1}) is given by

$$\begin{cases} x_{n+1} = x_0 x_n + d y_0 y_n, \quad x_1 = x_0, \quad n \geq 1 \\ \\ y_{n+1} = y_0 x_n + x_0 y_n, \quad y_1 = y_0, \quad n \geq 1 \end{cases} \tag{6}$$

Example 9

The equation $x^2 - 2y^2 = 1$, has the fund. sol. $(x_0, y_0) = (3{,}2)$. So

$$x_2 = x_0^2 + dy_0^2 = 9 + 2.4 = 17, \quad y_2 = y_0 x_0 + x_0 y_0 = 6 + 6 = 12$$

is also a solution: $17^2 - 2.12^2 = 1$.

General Solution of Pell's Equation

Theorem 10

Let Pell's equation $x^2 - dy^2 = 1$, have the fundamental solution (x_0, y_0). Then (x_n, y_n) is also a solution, given by

$$\begin{cases} x_n = \frac{1}{2}[(x_0 + y_0\sqrt{d})^n + (x_0 - y_0\sqrt{d})^n] \\ \\ y_n = \frac{1}{2\sqrt{d}}[(x_0 + y_0\sqrt{d})^n - (x_0 - y_0\sqrt{d})^n] \end{cases} \tag{7}$$

Example 11

Solve $x^2 - 2y^2 = 1$. The fund. sol. is (3,2). The general solution is:

$$x_n = \frac{1}{2}[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n], \quad y_n = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n]$$

The General Form of Pell's Equation

Definition 12

The general Pell's equation has the form

$$ax^2 - by^2 = 1 \tag{8}$$

where ab not a perfect square.

The equation

$$u^2 - abv^2 = 1 \tag{9}$$

is called the Pell's resolvent of Eq. (8)

The General Form of Pell's Equation

Theorem 13

Let

$$ax^2 - by^2 = 1$$

have an integral solution. Let (A, B) solution for least positive A, B. The general solution is

$$\begin{aligned} x_n &= Au_n + bBv_n \\ y_n &= Bu_n + aAv_n \end{aligned} \tag{10}$$

Where (u_n, v_n) is the general solution of Pell's resolvent $u^2 - abv^2 = 1$.

The General Form of Pell's Equation

Example 14

Solve

$$6x^2 - 5y^2 = 1 \tag{11}$$

The fund. sol. is $(x, y) = (A, B) = (1, 1)$. The resolvent is $u^2 - 30v^2 = 1$, with fund. sol. $(u_0, v_0) = (11, 2)$. The general solution of the resolvent is

$$\begin{cases} u_n = \dfrac{1}{2}[(11 + 2\sqrt{30})^n + (11 - 2\sqrt{30})^n] \\ \\ v_n = \dfrac{1}{2\sqrt{30}}[(11 + 2\sqrt{30})^n - (11 - 2\sqrt{30})^n] \end{cases}$$

The general solution of Eq. (11) is

$$x_n = u_n + 5v_n, \quad y_n = u_n + 6v_n$$

Training Problem 1

Problem 1

Find all integers $n \geq 1$ *such that* $2n + 1$ *and* $3n + 1$ *are both perfect squares.*

Training Problem 1

Problem 1

Find all integers $n \geq 1$ such that $2n+1$ and $3n+1$ are both perfect squares.

Observe that

$$2n+1 = x^2, 3n+1 = y^2 \implies 3x^2 - 2y^2 = 1,$$

with $3.2 = 6$ not a square in $\mathbb{Z}$.
So solving this amounts to solving the general form of Pell's equation.

The Negative Pell's Equation

Definition 15

The negative Pell's equation has the form

$$x^2 - dy^2 = -1 \tag{12}$$

where d not a perfect square.

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Theorem 16

Let (A, B) be the smallest positive solution to Eq. (12). Then the general solution to Eq. (12) is given by

$$\begin{cases} x_n = Au_n + dBv_n \\ y_n = Au_n + Bv_n \end{cases} \tag{13}$$

where (u_n, v_n) is the general solution of $u^2 - dv^2 = 1$.

Training Problem 2

Problem 2

Find all pairs (k, m) *such that*

$$1 + 2 + \cdots + k = (k + 1) + (k + 2) + \cdots + m.$$

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Adding $1 + 2 + \cdots + k$ to both sides of the above equality we get

$$2k(k + 1) = m(m + 1) \iff (2m + 1)^2 - 2(2k + 1)^2 = -1.$$

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Adding $1 + 2 + \cdots + k$ to both sides of the above equality we get

$$2k(k+1) = m(m+1) \iff (2m+1)^2 - 2(2k+1)^2 = -1.$$

The associated negative Pell's equation is $x^2 - 2y^2 = -1$ with the minimal solution $(A, B) = (1, 1)$.

Training Problem 3

Problem 3 (Romanian M. Olympiad, 1999)

Show that the equation $x^2 + y^3 + z^3 = t^4$ *has infinitely many solutions* $x, y, z, t, \in \mathbb{Z}$ *with the greatest common divisor 1.*

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Start from the equality

$$\begin{aligned} [1^3 + 2^3 + \cdots + (n-2)^3] + (n-1)^3 + n^3 &= \left(\frac{n(n+1)}{2}\right)^2 \\ \left[\frac{(n-2)(n-1)}{2}\right]^2 + (n-1)^3 + n^3 &= \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

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Do there exist infinitely many integers $n \geq 1$ such that $\frac{n(n+1)}{2}$ is a perfect square?

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\end{aligned}$$

Do there exist infinitely many integers $n \geq 1$ such that $\frac{n(n+1)}{2}$ is a perfect square?

$$\begin{aligned}
n(n+1) = n^2 + n = 2m^2 \quad &\iff \quad 4n^2 + 4n = 8m^2 \\
&\iff \quad (2n+1)^2 - 2(2m)^2 = 1
\end{aligned}$$

This is Pell's equation, which has infinitely many solutions.

Training Problem 4

Problem 4 (Irish M. Olympiad, 1995)

Determine all integers a such that the equation $x^2 + axy + y^2 = 1$ has infinitely many solutions.

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Rewrite the given equation in the form

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1. If $a^2 - 4 < 0$ then we have a finite number of solutions.

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2. If $a^2 - 4 = 0$ the equation becomes $2x + ay = \pm 2$ with infinitely many solutions.

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1. If $a^2 - 4 < 0$ then we have a finite number of solutions.
2. If $a^2 - 4 = 0$ the equation becomes $2x + ay = \pm 2$ with infinitely many solutions.
3. If $a^2 - 4 > 0$, then $a^2 - 4$ cannot be a perfect square and so the Pell's equation $u^2 - (a^2 - 4)v^2 = 1$ has infinitely many solutions. Letting $x = u - av, y = 2v$, we also have infinitely many solutions for $a^2 - 4 \geq 0$

Training Problem 5

Problem 5 (Bulgarian M. Olympiad, 1999)

Solve $x^3 = y^3 + 2y^2 + 1$ *for integers* x, y.

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Solve $x^3 = y^3 + 2y^2 + 1$ *for integers* x, y.

If $y^2 + 3y > 0$ then

$$y^3 < x^3 = y^3 + 2y^2 + 1 < (y^3 + 2y^2 + 1) + (y^2 + 3y) = (y + 1)^3,$$

which is impossible.

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If $y^2 + 3y > 0$ then

$$y^3 < x^3 = y^3 + 2y^2 + 1 < (y^3 + 2y^2 + 1) + (y^2 + 3y) = (y+1)^3,$$

which is impossible.
Therefore

$$y^2 + 3y \leq 0 \implies y = 0, -1, -2, -3.$$

The solution set is $(1, 0), (1, -2), (-2, -3)$.

Training Problem 6

Problem 6

Find positive integers x, y, z *such that* $xy + yz + zx - xyz = 2$

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We may assume that $x \leq y \leq z$.

1. If $x = 1$ then the equation is $y + z = 2 \implies (x, y, z) = (1, 1, 1)$

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2. If $x = 2$ then the equation is
   $2y + 2z - yz = 2 = (z - 2)(y - 2) \implies z = 4, y = 3.$

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3. If $x \geq 3$ then $x, y, z, \geq 3$ which yield

$$xyz \geq 3xy$$
$$xyz \geq 3yz$$
$$xyz \geq 3zx$$

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$$xyz \geq 3yz$$
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   Adding the above relations it follows that

$$xyz \geq xy + yz + zx \implies xy + yz + zx - xyz < 0 \neq 2.$$

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$$3^x = z^2 - 4^y = (z - 2^y)(z + 2^y).$$

Then

$$z - 2^y = 3^m \text{ and } z + 2^y = 3^n, \; m > n \geq 0, m + n = x.$$

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Then

$$z - 2^y = 3^m \text{ and } z + 2^y = 3^n,\ m > n \geq 0, m + n = x.$$

Subtracting,

$$\begin{aligned} 2^{y+1} &= 3^n - 3^m = 3^m(3^{n-m} - 1) \\ &\implies 3^m = 1, n = x \implies 3^n - 1 = 2^{y+1} \end{aligned}$$

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1. If $y = 0$, then $n = x = 1$ and $z = 2$.
2. If $y \geq 1$ then $x = n = 2, y = 2, z = 3^n - 2^y = 5$.

Training Problem 8

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Find the positive integers x, y, z *such that* $3^x - 1 = y^z$.

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Find the positive integers x, y, z *such that* $3^x - 1 = y^z$.

If z is even we get a contradiction. So $z = 2k + 1$.

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If z is even we get a contradiction. So $z = 2k + 1$. Now

$$3^x = y^z + 1 = y^{2k+1} + 1 = (y+1)(y^{2k} - y^{2k-1} + y^{2k-2} - \cdots + y^2 - y + 1).$$

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Then $y \equiv -1 \mod 3$.

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$3^x = y^z + 1 = y^{2k+1} + 1 = (y+1)(y^{2k} - y^{2k-1} + y^{2k-2} - \cdots + y^2 - y + 1)$.

Then $y \equiv -1 \mod 3$.

$$y^{2k} - y^{2k-1} + \cdots + y^2 - y + 1 \equiv \underbrace{1 + 1 + \cdots + 1}_{\text{2k }+1} \equiv (2k+1) \equiv 0 \mod 3.$$

Therefore $z = 2k + 1 = 3p$, some p:

$$3^x = y^{3p} + 1 = (y^p + 1)(y^{2p} - y^p + 1) \implies y^p + 1 = 3^s.$$

$$\begin{aligned}
3^x &= 1 + y^{3p} = 1 + (3^s - 1)^3 \\
&= 3^{3s} - 3.3^{2s} + 3.3^s \\
&= 3^{s+1}(3^{2s-1} - 3^5 + 1) \\
&\implies 3^{2s-1} - 3^s = 0 \implies s = 1 \\
&\implies y^p = 3^s - 1 = 2 \implies y = 2, p = 1, x = 2, z = 3.
\end{aligned}$$

Training Problem 9

Problem 9 (Taiwanese M. Olympiad, 1999)

Find all positive integers $a, b, c \geq 1$ *such that* $a^b + 1 = (a+1)^c$

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1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a+1)^c \equiv (-1)^b + 1 \equiv 0 \mod a+1 \implies b$ odd

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3. $(a+1-1)^b + 1 \equiv b(a+1) \equiv 0 \mod (a+1)^2 \implies a$ even

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4. $a^b + 1 \equiv 1 \equiv (a+1)^c \equiv ca+1 \mod a^2 \implies a|c \implies c$ even
5. $(2x)^b = (a+1)^{2y} - 1 = [(a+1)^y - 1][(a+1)^y + 1]$

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5. $(2x)^b = (a+1)^{2y} - 1 = [(a+1)^y - 1][(a+1)^y + 1]$
6. $gcd((a+1)^y - 1, (a+1)^y + 1) = 2$

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6. $gcd((a+1)^y - 1, (a+1)^y + 1) = 2$
7. $x|(a+1)^y - 1 = (2x+1)^y - 1 \implies (a+1)^y - 1 = 2x^b$

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4. $a^b + 1 \equiv 1 \equiv (a+1)^c \equiv ca+1 \mod a^2 \implies a|c \implies c$ even
5. $(2x)^b = (a+1)^{2y} - 1 = [(a+1)^y - 1][(a+1)^y + 1]$
6. $gcd((a+1)^y - 1, (a+1)^y + 1) = 2$
7. $x|(a+1)^y - 1 = (2x+1)^y - 1 \implies (a+1)^y - 1 = 2x^b$
8. $2^{b-1} = (a+1)^y + 1 > (a+1)^y - 1 = 2x^b \implies x = 1$

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Find all positive integers $a, b, c \geq 1$ *such that* $a^b + 1 = (a+1)^c$

1. $b = c = 1, a \geq 1$ is a solution. Let $b \geq 2$.
2. $a^b + 1 = (a+1)^c \equiv (-1)^b + 1 \equiv 0 \mod a+1 \implies b$ odd
3. $(a+1-1)^b + 1 \equiv b(a+1) \equiv 0 \mod (a+1)^2 \implies a$ even
4. $a^b + 1 \equiv 1 \equiv (a+1)^c \equiv ca + 1 \mod a^2 \implies a|c \implies c$ even
5. $(2x)^b = (a+1)^{2y} - 1 = [(a+1)^y - 1][(a+1)^y + 1]$
6. $gcd((a+1)^y - 1, (a+1)^y + 1) = 2$
7. $x|(a+1)^y - 1 = (2x+1)^y - 1 \implies (a+1)^y - 1 = 2x^b$
8. $2^{b-1} = (a+1)^y + 1 > (a+1)^y - 1 = 2x^b \implies x = 1$
9. The only other solution is $a = 2, b = c = 3$.